THE KRIGING OXYMORON: A CONDITIONALLY UNBIASED AND ACCURATE PREDICTOR (2nd EDITION)

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Abstract. An analysis of conditional bias and its impact on mineral resource estimation is presented. A simple method is proposed for building a long-term mineral resource block model that accounts for conditional bias, change-of-support, and the information effect at the time of mining.

1. Introduction

Accounting for change-of-support, the information effect, and conditional bias are problems well known to mineral resource modelers. Although the methods proposed for dealing with change-of-support and the information effect are little more than approximations, case studies suggest these methods are useful (David, 1977; Journel and Huijbregts, 1978; Matheron, 1984; Parker, 1980; Isaaks and Srivastava, 1991; Deraisme, 2000). However, the same cannot be said for conditional bias. A literature review reveals that conditional bias is poorly understood and that many of the claims are misleading.

Krige (1994; 1996; 1999) claims the preliminary prerequisite of all resource estimators is the elimination of conditional bias. Sinclair and Blackwell (2002) claim that conditional bias contributes to the discrepancies noted between the prediction of recoverable resources and production. David, Marcotte, and Soulie (1984) propose a correction for conditional bias and claim that this correction will reduce the discrepancy between predicted resources and production. Pan (1998) proposes a correction for conditional bias followed by a correction for the smoothing induced by the first correction. However, it can be shown that these two corrections are circular in the sense that the final smoothing correction re-introduces conditional bias. Guertin (1984) proposes a solution that she claims can be easily implemented as a correction factor for *any mineral resource estimation* or grade control system. Deutsch and McLennan (2003) argue that conditionally simulated block model values are both conditionally unbiased and accurate predictors of the tons and grade that will be recovered at the time of mining. However, as will be shown these claims are not correct despite their wide acceptance.

A conditionally unbiased and accurate predictor[1] is an oxymoron. The estimator for a long-term mine planning block model may be conditionally unbiased but then the histogram of block estimates will be smoothed yielding inaccurate predictions of the recoverable tons and grade above cutoff grade. Conversely, if the histogram of block estimates provides accurate predictions, then the block estimator is necessarily conditionally biased. The estimator for a long-term mine planning block model cannot be conditionally unbiased and simultaneously accurate as claimed by Deutsch and McLennan (2003). David (1977) recognized the oxymoron by pointing out that one can

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[1] Accuracy is defined as the ability of the long-term block model to predict the actual tonnage and average ore grade that will be recovered at the time of mining.

accurately estimate the histogram of block grades but then one cannot localize the blocks. Alternatively, one can estimate as accurately as possible the grades of precisely located blocks (thereby minimizing conditional bias) but then the block histogram will be smoothed. The only exception to this apparent contradiction occurs when the block estimates are perfectly correlated with the true block grades. In this unlikely scenario the block model is both a conditionally unbiased and accurate predictor.

This paper provides an analysis of conditional bias and its impact on mineral resource estimation. Although it may not be possible to eliminate conditional bias from the grade control estimator it can be evaluated through conditional simulation. Block models for long-term mine planning can be built using simulation methods that not only quantitatively account for the conditional bias of future grade control estimators, but also for future change-of-support and information effects.

Section 2 defines two types of mineral resource block models on the basis of how the block estimates are used by the mine. These definitions provide the key to understanding the role of conditional bias in mineral resource modeling. Section 3 provides a formal definition of conditional bias and describes a simple check. Section 4 examines the impact of conditional bias on prediction when the block estimates are used for selection at the time of mining e.g., grade control. Section 5 examines the problem of predicting the tons and grade that will be recovered at the time of mining given that selection will be made using grade control estimates based on future blast hole data. Section 6 describes how to build a long-term mine planning block model by conditional simulation that accounts for a future conditionally biased grade control estimator, the information effect, and a change of support.

2. Two Types of Block Models

Mineral resource models can be classified into one of two types depending on how the block estimates are used by the mine operation.

Type 1: Models whose block grade estimates are used to predict the tons and average grade of ore material that will be recovered each annual, semi-annual, or quarterly period over the life-of-mine are classified as Type 1. Individual block estimates are typically derived from relatively sparse diamond drill hole (DDH) data. Predicted recoveries made from Type 1 estimates are useful for feasibility studies, long and short term mine planning, and the estimation of production schedules etc. Individual block estimates are not used for selection at the time of mining. Thus, it is not necessary to know the precise location or recoverable grade of each ore block. Knowledge of the distribution of recoverable[2] block grades to be mined in the future for each period is sufficient. Type 1 models are often referred to as long-term (mine planning) models.

Type 2: Models whose block grade estimates are used for selection at the time of mining are classified as Type 2. Individual block volumes are equivalent to a selective mining unit (SMU) with the grade of each block typically estimated from neighboring blast hole (BH) grades. The use of these estimates to distinguish between ore and waste is commonly known as *grade control*.

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[2] The recoverable grade is the actual grade recovered given that selection is based on estimates typically made from blast hole data at the time of mining.

3. Definition of Conditional Bias

3.1 NOTATION

D : The deposit or domain of interest.
$[Z(\mathbf{u}),\ \mathbf{u} \in D]$: A stationary random function consisting of a set of point support random variables.

$Z_v(\mathbf{u}) = \frac{1}{|v|}\int_{v(\mathbf{u})} Z(\mathbf{u}')d\mathbf{u}'$: A random variable of support v centered at location $\mathbf{u}$.

$[Z_v(\mathbf{u}), \mathbf{u} \in D]$: A stationary random function consisting of a set of random variables of support *v*. The random function $[Z_v(\mathbf{u}), \mathbf{u} \in D]$ is written as Z_v to simplify notation.
$F_v(z;\mathbf{u} \mid (n)) = \text{prob}\{Z_v(\mathbf{u}) \le z \mid (n)\}$: Non stationary cumulative conditional distribution function (ccdf) of the random variable $Z_v(\mathbf{u})$ at the location $\mathbf{u}$ conditioned by n data.

$F_D(z; v \mid (n)) = \frac{1}{|D|}\int_D F_v(z;\mathbf{u}' \mid (n))d\mathbf{u}'$: The probability that the grade of a randomly selected SMU within the domain D will be no greater than the cutoff *z*.
$[Z_{v*}(\mathbf{u} \mid (n)), \mathbf{u} \in D]$: A non stationary random function consisting of a set of random variables where each RV $Z_{v*}(\mathbf{u} \mid (n))$ is of the form $\sum_{i=1}^{n} w_i\ Z(\mathbf{u}_i)$ with $\sum w = 1$. The random function $Z_{v*}(\mathbf{u} \mid (n)),\ \ \mathbf{u} \in D$ is written as Z_{v*} to simplify notation.
$F_{v*}(z;\mathbf{u} \mid (n)) = \text{prob}\{Z_{v*}(\mathbf{u} \mid (n)) \le z\}$: The non-stationary ccdf of the random variable $Z_{v*}(\mathbf{u})$ at the location $\mathbf{u}$ conditioned by the *(n)* data.

$F_D(z; v^* \mid (n)) = \frac{1}{|D|}\int_D F_{v^*}(z;\mathbf{u}' \mid (n))d\mathbf{u}'$: The probability that the estimated grade $Z_{v*}(\mathbf{u})$ of a SMU randomly selected within *D* will be no greater than *z*.

3.2 DEFINITION

The conditional expectation is given by:

$$E\{Z_v \mid Z_{v*} = z\} = h(z) \qquad \forall z \tag{1}$$

where the function $h(z)$ may be linear or non linear. However, if we impose the condition:

$$h(z) = z \qquad \forall z \tag{2}$$

the function $h(z)$ will be linear through the origin with a slope of 1.0 and the estimator Z_{v*} is conditionally unbiased by definition (Journel and Huijbregts, 1978).
The conditionally unbiased relation (2) can be re-written as:

$$E\{Z_v - Z_{v*} \mid Z_{v*} = z\} = 0 \qquad \forall z \tag{3}$$

Equations (2) and (3) imply that the average of the estimator Z_{v*} above cutoff is an unbiased estimate of the average of the corresponding true values Z_v :

$$E\{Z_v \mid Z_{v*} > z\} = E\{Z_{v*} \mid Z_{v*} > z\} \qquad \forall z \tag{4}$$

Equation (4) can also be written as:

$$E\{Z_v - Z_{v*} \mid Z_{v*} > z\} = 0 \qquad \forall z \tag{5}$$

3.3 A CHECK FOR CONDITIONAL BIAS

The linear regression of Z_v on Z_{v*} is given by $E\{Z_v \mid Z_{v*} = z\} = a * z + b$ where *a* is the slope and *b* the intercept. Thus, if the slope of the linear regression of Z_v on Z_{v*} is not equal to 1 or the intercept is not equal to 0, then the estimator Z_{v*} is conditionally biased, e.g.,

$$\begin{aligned} E\{Z_v \mid Z_{v*} = z\} &= h(z) \\ a * z + b &= z, \ \forall z \qquad \text{iff } a = 1 \text{ and } b = 0 \end{aligned} \tag{6}$$

The linear regression model also provides some insight on the relationship between the two random functions Z_v and Z_{v*} e.g., the slope *a* is given by:

$$a = \frac{\mathrm{cov}(Z_v Z_{v*})}{\mathrm{var}(Z_{v*})} = \frac{\sigma_v}{\sigma_{v*}} \rho_{v\,v*} \tag{7}$$

where σ_v^2 and σ_{v*}^2 are the variances of Z_v and Z_{v*}. Thus, for a conditionally unbiased estimator:

$$a = \frac{\sigma_v}{\sigma_{v*}} \rho_{vv*} = 1.0 \tag{8}$$

Two important observations can be made from Equation (8).

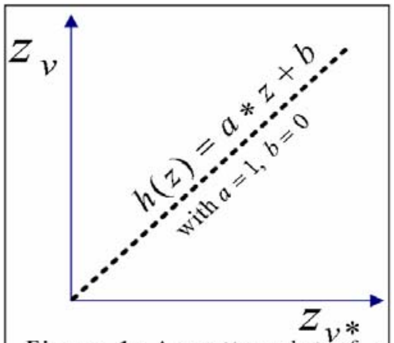

Figure 1: A scatter-plot of a conditionally unbiased estimator.

1. Since in practice, the correlation between the true and estimated values is: $\rho_{v\,v*} < 1$, then for a conditionally unbiased estimator, necessarily: $\sigma_v^2 > \sigma_{v*}^2$. In other words, the estimates of a conditionally unbiased estimator are smoothed.
2. Conversely, if the two distributions $F_D(z; v \mid (n))$ and $F_D(z; v* \mid (n))$ defined in section 3.1 have equal variances: $\sigma_v^2 = \sigma_{v*}^2$, then necessarily $a < 1$. That is, the estimator Z_{v*} is conditionally biased.

4. Type 2 Estimates and their Recovery Functions

Recall that the estimates of a Type 2 estimator are used by the mine operation for the selection of ore at the time of mining.

4.1 NOTATION

The type 2 estimator is denoted by a double asterisk, e.g., $Z_{v**}(\mathbf{u})$ **.**

$F_{v**}(z; \mathbf{u} \mid (n))$: The non stationary ccdf of the RV $Z_{v**}(\mathbf{u})$ at location $\mathbf{u}$ conditioned by the (n) data.

$F_D(z; v^{**} \mid (n)) = \frac{1}{|D|}\int_D F_{v^{**}}(z; \mathbf{u}' \mid (n)) d\mathbf{u}'$: The non stationary conditional probability that the estimated grade $Z_{v^{**}}(\mathbf{u})$ of a randomly selected SMU within the domain D will be no greater than z. Note that this distribution is commonly estimated in practise.

$F_v(z; \mathbf{u} \mid v^{**}, (n))$: The non stationary ccdf of the RV $Z_v(\mathbf{u})$ at location $\mathbf{u}$ given that $Z_{v^{**}}(\mathbf{u}) \le z$ and the (n) conditioning data.

$F_D(z; v \mid v^{**}, (n)) = \frac{1}{|D|}\int_D F_v(z; \mathbf{u}' \mid v^{**}, (n)) d\mathbf{u}'$: The non stationary conditional probability that the true grade $Z_v(\mathbf{u})$ of a randomly selected SMU within the domain D will be no greater than z given that its estimated grade $Z_{v^{**}}(\mathbf{u})$ is no greater than z. Note that this distribution is not known nor is it commonly estimated in practice.

4.2 ACTUAL RECOVERIES GIVEN THAT SELECTION IS MADE USING ESTIMATED GRADES.

The following recovery functions describe the *actual but unknown* quantities that will be recovered given that selection is made using the estimates $z_v^{**}(\mathbf{u})$**.** The recovered tonnage is given by:

$$T_D(z) = T_o[1 - F_D(z; v^{**} \mid (n))] \qquad \forall z \tag{9}$$

The actual but unknown quantity of recovered metal is given by:

$$Q_D(z) = T_o \int_z^{\infty} z' \, dF_D(z'; v \mid v^{**}, (n)) \qquad \forall z \tag{10}$$

The actual but unknown recovered grade is given by:

$$m_D(z) = \frac{Q_D(z)}{T_D(z)} \tag{11}$$

4.3 ESTIMATED RECOVERIES GIVEN THAT SELECTION IS MADE USING ESTIMATED GRADES

The recovery equations provided by (10) and (11) are not useful since the distribution $F_D(z; v \mid v^{**}, (n))$ is not known or commonly estimated in practice. However, by replacing the unknown distribution with the commonly estimated distribution $F_D(z; v^{**} \mid (n))$, one can estimate the recoveries as follows:

$$\hat{T}_D(z) = T_o[1 - F_D(z; v^{**} \mid (n))] \qquad \forall z \tag{12}$$

The estimated recovered quantity of metal is given by:

$$\hat{Q}_D(z) = T_o \int_z^{\infty} z' \, dF_D(z'; v^{**} \mid (n)) \qquad \forall z \tag{13}$$

and the estimated recovered grade is given by:

$$\hat{m}_D(z) = \frac{\hat{Q}_D(z)}{\hat{T}_D(z)} \tag{14}$$

If the estimator $Z_{v**}(\mathbf{u})$ is conditionally unbiased, then the estimated recoveries (13) (14) will be equal to the actual recoveries (10) (11) since conditional unbias implies the following:

$$
\begin{aligned}
& E\{Z_v \mid Z_{v**} > z\} && = E\{Z_{v**} \mid Z_{v**} > z\} \\
\Rightarrow\ & \frac{\int_z^\infty z' dF_D(z'; v \mid v^{**}; (n))}{1 - F_D(z; v^{**} \mid (n))} && = \frac{\int_z^\infty z' dF_D(z'; v^{**} \mid (n))}{1 - F_D(z; v^{**} \mid (n))} \qquad \forall z \qquad (15) \\
\Rightarrow\ & \int_z^\infty z' dF_D(z'; v \mid v^{**}; (n)) && = \int_z^\infty z' dF_D(z'; v^{**} \mid (n)) \\
\Rightarrow\ & F_D(z'; v \mid v^{**}; (n)) && = F_D(z'; v^{**} \mid (n))
\end{aligned}
$$

Thus, it appears[3] that the estimator $Z_{v**}(\mathbf{u})$ must be conditionally unbiased in order to provide accurate predictions of the tons and grade that will be delivered to the mill. Ideally, the estimator $Z_{v**}(\mathbf{u})$ will also minimize the conditional variance $E\{[Z_v - h(z)]^2\}$ (Journel and Huijbregts, 1978) so as to minimize ore loss and dilution or misclassification at the time of mining.

5. Type 1 Estimates and their Recovery Functions

Recall, that Type 1 estimates are used to predict the tons and grade of ore that will be recovered in the future at the time of mining. They are not used for selection at the time of mining.

5.1 NOTATION

Type 1 estimates are denoted by a single asterisk, e.g., $z_{v*}(\mathbf{u})$ **.**

$F_{v*}(z; \mathbf{u} \mid (n))$: The non stationary ccdf of the RV $Z_{v*}(\mathbf{u})$ at location $\mathbf{u}$ conditioned by the (n) data.

$F_D(z; v^* \mid (n)) = \frac{1}{|D|}\int_D F_{v*}(z; \mathbf{u}' \mid (n)) d\mathbf{u}'$: The non stationary conditional probability that the estimated grade $Z_{v*}(\mathbf{u})$ of a randomly selected SMU within the domain D will be no greater than z. Note that this distribution is commonly estimated in practice.

5.2 THE RECOVERY EQUATIONS

Recoverable tonnage:

$$\hat{T}_D(z) = T_o[1 - F_D(z; v^* \mid (n))] \qquad \forall z \qquad (16)$$

Recoverable quantity of metal:

$$\hat{Q}_D(z) = T_o \int_z^\infty z' \, dF_D(z'; v^* \mid (n)) \qquad \forall z \qquad (17)$$

[3] Section 6 describes how conditional simulation can be used to accurately predict the tons and grade that will be delivered to the mill in spite of a conditionally biased grade control estimator.

Recoverable grade:

$$\hat{m}_D(z) = \frac{\hat{Q}_D(z)}{\hat{T}_D(z)} \tag{18}$$

Recall, that (9), (10), and (11) provide the actual recoveries given that selection will be made using the estimates Z_{v**} in the future. Thus, to be useful the recoveries predicted by (16), (17), and (18) must be equal to those given by (9), (10), and (11). However, this is a problem since there is nothing in (16), (17), and (18) that guarantees equivalence to (9), (10), and (11). This problem is recognized within the mining industry where a common solution is to impose additional constraints on the estimators $z_{v*}(\mathbf{u})$ and $z_{v**}(\mathbf{u})$, e.g.,

Condition 1. $F_D(z;v^*|(n)) = F_D(z;v^{**}|(n'))$ - This condition requires the histogram of the type 1 estimates to be equal to the histogram of the type 2 estimates within *D*. For example;

- In practice, the future distribution $F_D(z;v^{**}|(n))$ is estimated using smoothing relations and the change of support hypothesis (Journel and Huijbregts, 1978; Isaaks and Srivastava, 1989; Sinclair and Blackwell, 2002).
- The distribution $F_D(z;v^*|(n))$ is then made to match as close as possible to the estimated distribution $F_D(z;v^{**}|(n))$ by controlling the number of samples used to estimate z_{v*} locally (Deutsch and McLennon, 2003).

Condition 2. $E\{Z_v - Z_v^{**} \mid Z_v^{**} > z\} = 0 \;\; \forall z$ - This condition requires the type 2 estimator to be conditionally unbiased.

The equivalence between the predicted recoveries (16), (17), and (18) given conditions (1) and (2) and the actual recoveries (9), (10), and (11) is easily confirmed.

However, condition (1) may not be that easy to impose on the estimator Z_{v*}. The change of support and information effect may render the distributions $F_D(z;v^*|(n))$ and $F_D(z;v^{**}|(n))$ incomparable. Thus, at best this practice amounts to nothing more than an approximation.

Condition (2) may also be difficult if not impossible to impose on the future estimator Z_{v**}. Although kriging is said to be a *conditionally unbiased estimator,* in reality it is conditionally unbiased if and only if the distribution of $Z(\mathbf{u})$ is normal and its mean $E\{Z(\mathbf{u})\}$ is known (David, 1977). The problem is that almost all distributions of $Z(\mathbf{u})$ in mining applications are non-normal with relatively large coefficients of variation and large coefficients of skew. Because of this, it is very difficult if not impossible for the mine operator to insure that the grade control estimator is conditionally unbiased.

5.3 THE OXYMORON

Note, that although the estimator Z_{v^*} is an accurate predictor of recoveries (9), (10), and (11) given conditions (1) and (2), Z_{v^*} is almost certain to be conditionally biased. For example, from condition (2),

$$\frac{\sigma_v}{\sigma_{v^{**}}}\rho_{vv^{**}} = 1.0 \tag{19}$$

and from condition (1),

$$\sigma_{v^*} = \sigma_{v^{**}} \tag{20}$$

and since $\rho_{vv^*} < \rho_{vv^{**}}$ with near certainty then,

$$a = \frac{\sigma_v}{\sigma_{v^*}}\rho_{vv^*} < 1.0 \tag{21}$$

that is, Z_{v^*} is almost certain to be conditionally biased. Thus, in spite of conditional bias, Z_{v^*} may be an accurate predictor of recoverable resources given conditions (1) and (2).

6 Conditional Simulation and Prediction

This section proposes a method for building the long-term block model using conditional simulation via the LU decomposition of the covariance matrix, (Davis, 1987).

6.1 NOTATION

$Z_{\tilde{v}}(\mathbf{u})$ - the tilde above a variable denotes a conditionally simulated value. Otherwise the notation for the simulated variables and their distributions is identical to the definitions provided in section 4.1

6.2 CONDITIONAL SIMULATION

Consider the following vectors of point support Gaussian random variables:

$\mathbf{Y}_1 = [Y(\mathbf{u}_i), i = 1, n]'$ - a vector of (n) $N(0,1)$ random variables located at DDH sample locations $\mathbf{u}_i$ $i = 1, n$,
$\mathbf{Y}_2 = [Y(\mathbf{u}_j), j = 1, s]'$ - a vector of (s) $N(0,1)$ random variables located at blast hole (BH) locations $\mathbf{u}_j$ $j = 1, s$, and
$\mathbf{Y}_3 = [Y(\mathbf{u}_k), k = 1, t]'$ - a vector of (t) $N(0,1)$ random variables located at the discretization point locations $\mathbf{u}_k$ $k = 1, t$ of the SMU.
Note that some of the locations may be co-located, e.g., $\mathbf{u}_i = \mathbf{u}_j$, $\mathbf{u}_i = \mathbf{u}_k$, $\mathbf{u}_j = \mathbf{u}_k$ for some i, j, k (see Figure 2).
The corresponding covariance matrices are given by:

$\mathbf{C}_{11} = \text{cov}(\mathbf{Y}_1\mathbf{Y}_1')$ with dimension $n \ x \ n$

$\mathbf{C}_{21} = \text{cov}([\mathbf{Y}_2', \mathbf{Y}_3']'\mathbf{Y}_1')$ with dimension $m \ x \ n$ where $s + t = m$.

$\mathbf{C}_{22} = \text{cov}([\mathbf{Y}_2', \mathbf{Y}_3']'[\mathbf{Y}_2', \mathbf{Y}_3'])$ with dimension $m \ x \ m$.

The covariance matrix between the random vectors $\mathbf{Y}_1, \mathbf{Y}_2$, and $\mathbf{Y}_3$ can be decomposed into the product of a lower and upper triangular matrix, e.g.,

$$\left[\begin{array}{c|c} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \hline \mathbf{C}_{21} & \mathbf{C}_{22} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{L}_{11} & \mathbf{0} \\ \hline \mathbf{L}_{21} & \mathbf{L}_{22} \end{array}\right] * \left[\begin{array}{c|c} \mathbf{U}_{11} & \mathbf{U}_{21} \\ \hline \mathbf{0} & \mathbf{U}_{22} \end{array}\right] \tag{22}$$

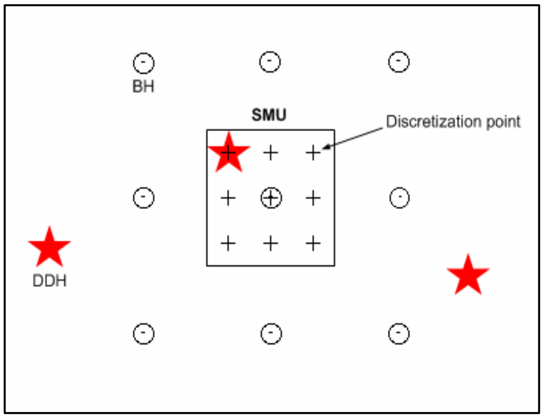

Figure2: Example locations of the random variables $\mathbf{Y}$ relative to a SMU. The stars represent $\mathbf{Y}_1$ at DDH locations, while the circles represent $\mathbf{Y}_2$ at BH locations and the plus signs symbolize $\mathbf{Y}_3$ at the discretization points of the SMU. Note the co-location of some of the variable locations.

Next, we interpret the relatively sparse DDH data $z(\mathbf{u}_i),\ i=1,n$ as a realization of the random vector $\mathbf{Y}_1$, e.g.,

$$y_1(\mathbf{u}_i) = \varphi(z(\mathbf{u}_i)),\ i=1,n \tag{23}$$

where $\varphi(\cdot)$ is the normal score transform. Realizations of the random vectors $\widetilde{\mathbf{Y}}_2$ (at BH locations) and $\widetilde{\mathbf{Y}}_3$ (at SMU discretization point locations) can be simulated conditional to the transformed DDH data $\mathbf{Y}_1$ as follows:

$$\begin{aligned} \begin{bmatrix} \mathbf{Y}_1 \\ \hline \widetilde{\mathbf{Y}}_2 \\ \widetilde{\mathbf{Y}}_3 \end{bmatrix} &= \begin{bmatrix} \mathbf{L}_{11} & \mathbf{0} \\ \hline \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} * \begin{bmatrix} \mathbf{L}_{11}^{-1}\mathbf{Y}_1 \\ \hline \mathbf{W} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{L}_{21}\mathbf{L}_{11}^{-1}\mathbf{Y}_1 + \mathbf{L}_{22}\mathbf{W} \end{bmatrix} \end{aligned} \tag{24}$$

where $\mathbf{W}$ is a random vector of (m) *iid* $N(0,1)$ random variables. Multiple realizations of the vectors $\widetilde{\mathbf{Y}}_2$ and $\widetilde{\mathbf{Y}}_3$ each conditional to $\mathbf{Y}_1$ (and to each other) are obtained by generating realizations of the *iid* random vector $\mathbf{W}$ and evaluating,

$$\begin{bmatrix} \widetilde{\mathbf{Y}}_2 \\ \widetilde{\mathbf{Y}}_3 \end{bmatrix} = \mathbf{L}_{21}\mathbf{L}_{11}^{-1}\mathbf{Y}_1 + \mathbf{L}_{22}\mathbf{W} \tag{25}$$

for each realization of $\mathbf{W}$. A single conditional simulation of the SMU grade at location $\mathbf{u}_0$ is given by:

$$z_{\tilde{v}}(\mathbf{u}_0) = \frac{1}{t}\sum_{k=1}^{t} \varphi^{-1}(\widetilde{y_3}(\mathbf{u}_k)) \tag{26}$$

The corresponding estimated SMU grade made from conditionally simulated blast hole grades is given by:

$$z_{\tilde{v}^{**}}(\mathbf{u}_0) = \sum_{j}^{s} \lambda_j B[\varphi^{-1}(\widetilde{y_2}(\mathbf{u}_j))\ ] \tag{27}$$

where λ are ordinary kriging weights for example and $\varphi^{-1}[\widetilde{\mathbf{Y}}_2]$ are simulated DDH values at the blast hole locations. $B[\cdot]$ is a user defined function for transforming simulated DDH grades to simulated BH grades. For example, the function $B[\cdot]$ could be used to add noise or deviations to the vector of simulated DDH grades $\varphi^{-1}[\widetilde{\mathbf{Y}}_2]$ (Parker and Isaaks, 1992; Journel and Kyriakidis, 2004).

The distributions $F_{\tilde{v}}(z;\mathbf{u}_0 \mid (n))$ and $F_{\tilde{v}}(z;\mathbf{u}_0 \mid \tilde{v}^{**},(n))$ of the conditionally simulated values $z_{\tilde{v}}(\mathbf{u}_0)$ and $z_{\tilde{v}^{**}}(\mathbf{u}_0)$ are generated by repeated applications of (25), (26), and (27). For example by using an efficient LU algorithm it may be practical to simulate as many as 500 equi-probable pairs of $z_{\tilde{v}}(\mathbf{u}_0)$ and $z_{\tilde{v}^{**}}(\mathbf{u}_0)$ for each SMU.

Thus, the simulated tonnage recovered over the domain D is given by:

$$\widetilde{T}_D(z) = T_o[1 - F_D(z;\tilde{v}^{**} \mid (n))] \qquad \forall z \tag{28}$$

The simulated actual quantity of recovered metal is given by:

$$\widetilde{Q}_D(z) = T_o \int_z^{\infty} z'\, dF_D(z';\tilde{v} \mid \tilde{v}^{**},(n)) \qquad \forall z \tag{29}$$

The simulated actual recovered grade is given by:

$$\widetilde{m}_D(z) = \frac{\widetilde{Q}_D(z)}{\widetilde{T}_D(z)} \tag{30}$$

Equation (26) solves the change of support problem by computing a simple spatial average from a number of jointly simulated point values within the SMU. Note that each simulated point value is back-transformed before averaging.

Equation (27) provides a simulation of the grade control estimator using simulated blast hole grades. Note, that (27) includes a user-definable function enabling the user to simulate the relationship between the DDH and BH grades if known (Parker and Isaaks, 1992; Journel and Kyriakidis, 2004). Thus, the impact of poorer quality blast hole assays on the predicted recoveries can be put into the estimation of recoverable resources here.
Equations (29) and (30) simulate the actual recovered quantity of metal and recovered grade given that the SMU are selected by their grade control estimate. The key is the simulated conditional distribution of the true SMU grades given their grade control estimates. This distribution quantitatively accounts for any conditional bias inherent in the grade control estimator as well as for any associated misclassification.

7 Conclusions

- If the block estimates are to be used for selection (grade control), then it is desirable to minimize conditional bias. Although conditional bias may be minimized, it likely cannot be eliminated.
- If the grade control estimator is conditionally biased, the predictions of the long-term mine planning model should quantitatively account for the bias. Such an accounting can be evaluated through conditional simulation.
- If the block estimates are not used for selection at the time of mining, but rather for the prediction of the tons and grade that will be recovered in the future, then whether or not the block estimator is conditionally biased is irrelevant to the accuracy of predicting the future recoveries.
- The predictions of the long-term model should quantitatively account for the ore loss and dilution (misclassification) that will occur at the time of mining. Again, such an accounting can be evaluated through conditional simulation.
- And finally, conditional simulation provides an easy solution to change of support. Long-term mine planning models with block sizes equivalent to the SMU are easily simulated. With good software, conditional simulation via the LU decomposition of the covariance matrix is as practical as ordinary kriging.

8 References

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